A multi-layer extension of the stochastic heat equation

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Abstract. Motivated by recent developments on solvable directed polymer models, we define a 'multi-layer' extension of the stochastic heat equation involving non-intersecting Brownian motions.

1. Introduction

We consider the stochastic heat equation in one dimension

$$\partial_t \mathcal{Z} = \frac{1}{2}\partial_y^2 \mathcal{Z} + \dot{W}(t,y)\mathcal{Z} \tag{1}$$

with initial condition $\mathcal{Z}(0,x,y) = \delta(x-y)$, where $\dot{W}(t,y)$ is space-time white noise [**15, 38**]. The solution $\mathcal{Z}(t,x,y)$ is given by the chaos expansion

$$\begin{aligned}\mathcal{Z}(t,x,y) &= p(t,x,y) + \\ &\sum_{k=1}^{\infty}\int_{\Delta_k(t)}\int_{\mathbb{R}^k} p(t_1,x,x_1)p(t_2-t_1,x_1,x_2)\cdots p(t-t_k,x_k,y) \\ &\qquad\qquad \times W(dt_1,dx_1)\cdots W(dt_k,dx_k),\end{aligned} \tag{2}$$

where $\Delta_k(t) = \{0 < t_1 < \cdots < t_k < t\}$ and

$$p(t,x,y) = \frac{1}{\sqrt{2\pi t}} e^{-(x-y)^2/2t}.$$

For each $t > 0$ and $x, y \in \mathbb{R}$, the expansion (2) is convergent in $L_2(W)$. It satisfies (1) in the sense that it satisfies the integral equation

$$\mathcal{Z}(t,x,y) = p(t,x,y) + \int_0^t \int_{\mathbb{R}} p(t-s,y',y)\mathcal{Z}(s,x,y')W(ds,dy'), \tag{3}$$

which can be seen by iterating this equation.

This solution arises as a scaling limit of discrete directed polymer models, in the 'intermediate disorder' regime [**1, 19**]. It has also been known for some time that $h = \log \mathcal{Z}$ arises as the scaling limit of the height profile of the weakly asymmetric simple exclusion process [**6**]. With this 'surface growth' interpretation,

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h is understood to be the physically relevant solution (also known as the Cole-Hopf solution) to the KPZ equation [**16**]

$$\partial_t h = \frac{1}{2}\partial_y^2 h + \frac{1}{2}(\partial_y h)^2 + \dot{W}(t,y),$$

with 'narrow wedge' initial condition.

In a remarkable recent development [**2, 7, 10, 11, 12, 28, 29, 30, 31**] the exact distribution of $\mathcal{Z}(t,x,y)$ has been determined. The results so far have been based on two distinct approaches. One is to use the asymmetric simple exclusion process approximation together with recent work by Tracy and Widom [**34, 35, 36, 37**] in which exact formulas have been obtained for that process using the Bethe ansatz. The other is based on replicas, where the moments of the partition function are related to the attractive δ-Bose gas and also computed using the Bethe ansatz. These developments indicate that there is an underlying 'integrable' structure behind KPZ and the stochastic heat equation which is not yet fully understood.

It has recently been found that there exist exactly solvable discrete directed polymer models [**9, 20, 22, 23, 32, 33**], yielding yet another approach. We will describe one of the main results from [**22**], which provides the motivation for the present work. Define an 'up/right path' in $\mathbb{R}\times\mathbb{Z}$ to be an increasing path which either proceeds to the right or jumps up by one unit. For each sequence $0 < s_1 < \cdots < s_{N-1} < t$ we can associate an up/right path ϕ from $(0,1)$ to (t,N) which has jumps between the points (s_i,i) and $(s_i,i+1)$, for $i=1,\ldots,N-1$, and is continuous otherwise. Let $B(t)=(B_1(t),\ldots,B_N(t))$, $t\geq 0$, be a standard Brownian motion in $\mathbb{R}^N$ and define

$$Z^N(t) = \int e^{E(\phi)} d\phi,$$

where

$$E(\phi) = B_1(s_1) + B_2(s_2) - B_2(s_1) + \cdots + B_N(t) - B_N(s_{N-1})$$

and the integral is with respect to Lebesgue measure on the Euclidean set of all such paths. This is the partition function for the model. In [**22**] a formula is given for the Laplace transform of the distribution of $Z^N(t)$, which is obtained via the following 'multi-layer' construction. For $n=1,2,\ldots,N$, define

$$Z_n^N(t) = \int e^{E(\phi_1)+\cdots+E(\phi_n)} d\phi_1 \ldots d\phi_n,$$

where the integral is with respect to Lebesgue measure on the Euclidean set of n-tuples of non-intersecting (disjoint) up/right paths with respective initial points $(0,1),\ldots,(0,n)$ and respective end points $(t,N-n+1),\ldots,(t,N)$. Define $X_1^N(t) = \log Z_1^N(t)$ and, for $n\geq 2$, $X_n^N(t) = \log[Z_n^N(t)/Z_{n-1}^N(t)]$. The relevance of this construction is analogous to the role of the RSK correspondence in the study of last passage percolation and longest increasing subsequence problems; in this setting it is based on a geometric (or 'tropical') variant of the RSK correspondence. The main result in [**22**] is that the process $X^N(t) = (X_1^N(t),\ldots,X_N^N(t))$, $t>0$, is a diffusion process in $\mathbb{R}^N$ with infinitesimal generator given by

$$\frac{1}{2}\Delta + \nabla\log\psi_0\cdot\nabla$$

where $\psi_0(x)$ is the ground state eigenfunction of the quantum Toda lattice Hamiltonian

$$H = \Delta - 2\sum_{i=1}^{N-1} e^{x_{i+1}-x_i}.$$

The law of the partition function $Z^N(t)$ is obtained as a corollary. This result has been extended to a discrete-time framework in [**9**], where it is related to the solvability of a lattice directed polymer model with log-gamma weights introduced by Seppalainen [**32**], also involves the eigenfunctions of the quantum Toda lattice (known as Whittaker functions) and works directly in the setting of the 'tropical RSK correspondence' introduced and studied in the papers [**18, 21**]. These results suggest that the continuum versions of the partition functions $Z_n^N(t)$, which we introduce in this paper, will play a role in our understanding of the 'integrable' structure which appears to lie behind KPZ and the stochastic heat equation.

The outline of the paper is as follows. In the next section, we define the continuum versions of the above partition functions as chaos expansions involving non-intersecting Brownian motions. In section 3, we study the analogue of the partition functions when the space-time white noise is replaced by a smooth time-varying potential. In this setting we establish a connection with Darboux transformations of solutions to the heat equation, which give rise to evolution equations for the multi-layer process of partition functions. These equations are not directly meaningful in the white noise setting, but suggest that the multi-layer process has a Markovian evolution. There is a multi-dimensional version of the stochastic heat equation based on Brownian motion in a Weyl chamber, for which a Markovian evolution is readily apparent, and in section 4, we establish an analogue of the Karlin-McGregor formula for the solution of this equation. In section 5 we explain how, using this formula, the multi-dimensional process can (in principle—we only prove it for the first two layers) be expressed in terms of the multi-layer process, thus explaining the Markov property of the latter. We conclude in section 6 by formulating a continuum version of the tropical RSK correspondence.

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2. The continuum partition functions

In this section we define the continuum analogues of the partition functions described in the introduction. For $n = 1, 2, \ldots$, $t \geq 0$ and $x, y \in \mathbb{R}$, define

$$\mathcal{Z}_n(t,x,y) = p(t,x,y)^n \Big(1 + \sum_{k=1}^{\infty} \int_{\Delta_k(t)} \int_{\mathbb{R}^k} R_k^{(n)}((t_1,x_1),\ldots,(t_k,x_k)) \times W(dt_1,dx_1)\cdots W(dt_k,dx_k)\Big), \tag{4}$$

where $R_k^{(n)}$ is the k-point correlation function for a collection of n non-intersecting Brownian bridges which all start at x at time 0 and all end at y at time t. Note that $\mathcal{Z}_1 = \mathcal{Z}$ is the solution of the stochastic heat equation defined by (2).

Theorem 2.1. *The series (4) is convergent in $L_2(W)$.*

PROOF. We need to show that

$$\sum_{k=0}^{\infty}\frac{1}{k!}\int_{[0,t]^k}\int_{\mathbb{R}^k} R_k^{(n)}((t_1,x_1),\ldots,(t_k,x_k))^2 dx_1\cdots dx_k dt_1\cdots dt_k<\infty. \tag{5}$$

This is equivalent to showing that $\mathbb{E}e^L<\infty$, where L is the total intersection local time between two independent copies $X=(X_s^i,\ 0\le s\le t,\ i=1,\ldots,n)$ and $Y=(Y_s^i,\ 0\le s\le t,\ i=1,\ldots,n)$ of the system of n non-intersecting Brownian bridges which all start at x at time 0 and all end at y at time t. We will show that, in fact, all exponential moments of L are finite. Without loss of generality we can assume that $x=y$. First note that $L=A+B$, where A is the intersection local time on the time interval $[0,t/2]$ and B is the remainder; by symmetry, A and B have the same distribution. Thus, by Cauchy-Schwartz, it suffices to show that A has finite exponential moments of all orders. Now, on the time interval $[0,t/2]$, X and Y are (up to a bounded Radon-Nikodym density and time-change) two independent copies of Dyson Brownian motion in Λ_n started at $\hat{x}$. It therefore suffices to show that, for two independent Dyson Brownian motions in Λ started at the origin and run for time T, say, the total intersection local time has finite exponential moments of all orders. Denote these two Dyson Brownian motions by U and V. Although these processes are necessarily defined via an entrance law (as they are started on the boundary of the Weyl chamber), it can be shown that they satisfy a system of SDEs

$$dU_s^i=d\beta_s^i+\sum_{j\ne i}\frac{ds}{U_s^i-U_s^j},\qquad dV_s^i=d\gamma_s^i+\sum_{j\ne i}\frac{ds}{V_s^i-V_s^j},$$

where $\beta^i,\gamma^i,i=1,2,\ldots,n$ are a collection of independent standard one-dimensional Brownian motions. The total intersection local time up to time T is given by $\sum_{i\ne j}L^{ij}$, where L^{ij} denotes the local time (at zero) of U^i-V^j up to time T. Again by Cauchy-Schwartz, it suffices to show that for each distinct pair i,j, L^{ij} has finite exponential moments of all orders. In the following we will use the fact that the random variables $|U_T^i|$ (for each i) and the absolute value of a Gaussian random variable all have finite exponential moments of all orders.

By Tanaka's formula,

$$\begin{aligned}L^{ij} &= |U_T^i-V_T^j|-\int_0^T \mathrm{sgn}(U_s^i-V_s^j)d(U_s^i-V_s^j)\\ &= |U_T^i-V_T^j|-\int_0^T \mathrm{sgn}(U_s^i-V_s^j)d(\beta_s^i-\gamma_s^j)-\int_0^T \mathrm{sgn}(U_s^i-V_s^j)(D_s^i-E_s^j)ds\\ &\le |U_T^i-V_T^j|+\left|\int_0^T \mathrm{sgn}(U_s^i-V_s^j)d(\beta_s^i-\gamma_s^j)\right|+\int_0^T|D_s^i|ds+\int_0^T|E_s^j|ds,\end{aligned}$$

where

$$D_s^i=\sum_{j\ne i}\frac{1}{U_s^i-U_s^j},\qquad E_s^j=\sum_{j\ne i}\frac{1}{V_s^i-V_s^j}.$$

Thus, it suffices to show that each of the random variables

$$|U_T^i-V_T^j|,\qquad \left|\int_0^T \mathrm{sgn}(U_s^i-V_s^j)d(\beta_s^i-\gamma_s^j)\right|,$$

$$\int_0^T |D_s^i| ds \qquad \text{and} \quad \int_0^T |E_s^j| ds,$$

has finite exponential moments of all orders. For the first, we note that

$$|U_T^i - V_T^j| \leq |U_T^i| + |V_T^j|,$$

which has finite exponential moments of all orders. The second is the absolute value of a Gaussian random variable with mean zero and variance $2T$. So it remains to show that, for each i,

$$\xi_i := \int_0^T |D_s^i| ds$$

has finite exponential moments of all orders. We will prove this by induction over i. For $i < j$, define

$$\xi_{ij} = \int_0^T \frac{1}{U_s^i - U_s^j} ds.$$

First we note that

$$\xi_1 = \int_0^T D_s^i ds = U_T^1 - \beta_T^1,$$

has finite exponential moments of all orders. Now, since

$$\xi_1 = \xi_{12} + \cdots + \xi_{1n}$$

and each term is non-negative, this implies that $\xi_{1j} \leq \xi_1$ and hence has finite exponential moments of all orders for each $j = 2, \ldots, n$. Now

$$\xi_2 = \xi_{12} + \xi_{23} + \cdots + \xi_{2n} = \int_0^T D_s^2 ds + 2\xi_{12} = U_T^2 - \beta_T^2 + 2\xi_{12}.$$

Thus ξ_2 and $\xi_{23}, \ldots, \xi_{2n}$ all have finite exponential moments of all orders. Similarly,

$$\xi_3 = \xi_{13} + \xi_{23} + \xi_{34} + \cdots + \xi_{3n} = \int_0^T D_s^3 ds + 2\xi_{13} + 2\xi_{23} = U_T^3 - \beta_T^3 + 2\xi_{13} + 2\xi_{23},$$

and so on. $\square$

3. Darboux transformations and non-intersecting Brownian motions

In this section we replace the white noise potential by a smooth potential ϕ, which we assume for convenience to be in the Schwartz space E of rapidly decreasing smooth (C^∞) functions on $\mathbb{R}_+ \times \mathbb{R}$.

For each $n = 1, 2, \ldots$, $t > 0$ and $x, y \in \mathbb{R}$, define

$$Z_n(t,x,y) = p(t,x,y)^n \mathbb{E} \exp\left(\sum_{i=1}^n \int_0^t \phi(s, X_s^i) ds \right), \tag{6}$$

where X_s^i, $0 \leq s \leq t$, $i = 1, \ldots, n$ denote the trajectories of n non-intersecting Brownian bridges which all start at x at time 0 and all end at y at time t. On one hand, these are the analogues of the partition functions $\mathcal{Z}_n$'s introduced in the previous section with the white noise replaced by a smooth potential. On the other, they are directly related to the $\mathcal{Z}_n$'s by the formula

$$Z_n(t,x,y) = \mathbb{E}\left[\mathcal{Z}_n(t,x,y) \exp^\diamond(W(\phi))\right], \tag{7}$$

where $\exp^\diamond(W(\phi))$ is the Wick exponential of $W(\phi)$ defined by

$$\exp^\diamond(W(\phi)) = \exp\left(\int_0^\infty \int_R \phi(s,x) W(ds,dx) - \frac{1}{2}\int_0^\infty \int_R \phi(s,x)^2 dxds\right).$$

In other words, as a function of ϕ, $Z_n(t,x,y)$ is the S-transform of $\mathcal{Z}_n(t,x,y)$ [**15**]. To see that (7) holds, on the RHS replace $\mathcal{Z}_n(t,x,y)$ by the series (4) and $\exp^\diamond(W(\phi))$ by its Wiener chaos expansion; computing the expectation of the product of these two series we obtain

$$\begin{aligned} p(t,x,y)^n \sum_{k=0}^\infty \int_{\Delta_k(t)} \int_{R^n} \phi(t_1,x_1)\dots\phi(t_k,x_k) R_k^{(n)}((t_1,x_1),\dots(t_k,x_n)) dx_1\dots dx_k dt_1 \dots dt_k \\ = p(t,x,y)^n \mathbb{E} \exp\left(\sum_{i=1}^n \int_0^t \phi(s,X_s^i)ds\right). \end{aligned}$$

By the Feyman-Kac formula, $Z := Z_1$ satisfies the heat equation

$$\partial_t Z = \frac{1}{2}\partial_y^2 Z + \phi(t,y) Z \tag{8}$$

with initial condition $Z(0,x,y) = \delta(x-y)$.

Proposition 3.1.

$$Z_n(t,x,y) = c_{n,t} \det\left[\partial_x^i \partial_y^j Z(t,x,y)\right]_{i,j=0}^{n-1}, \tag{9}$$

where $c_{n,t} = t^{n(n-1)/2} \prod_{j=1}^{n-1} j!$.

We will prove this via a generalisation of the Karlin-McGregor formula. Set

$$\Lambda_n = \{x \in \mathbb{R}^n : \ x_1 \geq \cdots \geq x_n\}. \tag{10}$$

For each $t > 0$ and $x, y \in \Lambda_n^\circ$, define

$$\tilde{Z}_n(t,x,y) = p_n^*(t,x,y) \mathbb{E} \exp\left(\sum_{i=1}^n \int_0^t \phi(s,U_s^i)ds\right), \tag{11}$$

where U is a collection of non-intersecting Brownian bridges started at positions $x_1,\dots,x_n$ and ending at $y_1,\dots,y_n$, and $p_n^*(t,x,y)$ is the transition density of a Brownian motion in Λ_n killed when it hits the boundary, given by the Karlin-McGregor formula [**17**],

$$p_n^*(t,x,y) = \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) \prod_{i=1}^n p(t,x_i,y_{\sigma(i)}).$$

Proposition 3.2.

$$\tilde{Z}_n(t,x,y) = \det[Z(t,x_i,y_j)]_{i,j=1}^n. \tag{12}$$

Proof. According to the Feynman-Kac formula, $\tilde{Z}_n$ satisfies the equation

$$\partial_t \tilde{Z}_n = \frac{1}{2}\Delta_y \tilde{Z}_n + \sum_i \phi(t,y_i)\tilde{Z}_n \tag{13}$$

with Dirichlet boundary conditions on $\partial\Lambda_n$ and initial condition $\tilde{Z}_n(0,x,y) = \prod_i \delta(x_i - y_i)$. Moreover it is the unique solution to this initial-boundary value problem which vanishes as $|y| \to \infty$ uniformly for t in compact intervals.

On the other hand, $\det[Z(t,x_i,y_j)]_{i,j=1}^n$ satisfies the same initial-boundary value problem and vanishes as $|y|\to\infty$ uniformly for t in compact intervals. So the identity follows by uniqueness. □

Proof of Proposition 3.1. For $a\in\mathbb{R}$, denote by $\hat{a}\in\mathbb{R}^n$ the vector with all coordinates equal to a. Now, it is immediate from the definitions that

$$\frac{Z_n(t,a,b)}{p(t,a,b)^n}=\lim_{x\to\hat{a},y\to\hat{b}}\frac{\tilde{Z}_n(t,x,y)}{p_n^*(t,x,y)}.$$

By Proposition 3.2 we have

$$\lim_{x\to\hat{a},y\to\hat{b}}\frac{\tilde{Z}_n(t,x,y)}{\Delta(x)\Delta(y)}=\det\left[\partial_a^i\partial_b^j Z(t,a,b)\right]_{i,j=0}^{n-1},$$

where $\Delta(x)=\prod_{i<j}(x_i-x_j)$ (see, for example, [**39**]). On the other hand,

$$\lim_{x\to\hat{a},y\to\hat{b}}\frac{p_n^*(t,x,y)}{\Delta(x)\Delta(y)}=p(t,a,b)^n/c_{n,t},$$

which completes the proof. □

Define $u_n(t,x,y)$ recursively by $Z_n=u_1u_2\cdots u_n$.

Proposition 3.3. *The functions u_n satisfy the coupled system of heat equations*

$$\partial_t u_n=\frac{1}{2}\partial_y^2 u_n+[\phi(t,y)+\partial_y^2\log\left(Z_{n-1}/p^{n-1}\right)]u_n \tag{14}$$

with initial conditions $u_n(0,x,y)=\delta(x-y)$.

Proof. The equations follow from Proposition 3.2 together with known properties of Darboux transformations of solutions to one-dimensional heat equations with time-varying potentials, see for example [**3**]. The initial condition follows immediately from the definition of Z_n. □

The coupled heat equations of Proposition 3.3 are not immediately meaningful if we replace the smooth potential ϕ by space-time white noise. However they do suggest that the multi-layer process

$$(\mathcal{Z}_1(t,x,\cdot),\dots,\mathcal{Z}_n(t,x,\cdot)),\ t\geq 0$$

is Markov. In the following, we introduce a natural extension of the multi-layer process which will play an important role in our understanding of the Markov property when we return to the white noise setting.

Define, for $t>0$ and $x,y\in\Lambda_n$,

$$\hat{Z}_n(t,x,y)=\frac{\tilde{Z}_n(t,x,y)}{\Delta(x)\Delta(y)}. \tag{15}$$

This extends continuously to the boundary of $\Lambda_n\times\Lambda_n$; by Proposition 3.2, for $x\in\mathbb{R}$,

$$\hat{Z}_n(t,\hat{x},y)=\Delta(y)^{-1}\det\left[\partial_x^{i-1}Z(t,x,y_j)\right]_{i,j=1}^n. \tag{16}$$

Rather surprisingly, we will now show that the apparently richer object $\hat{Z}_n(t,\hat{x},\cdot)$ is, for a fixed $x\in\mathbb{R}$ and $t>0$, given as a function of $(Z_1(t,x,\cdot),\dots,Z_n(t,x,\cdot))$.

For notational convenience, set $Z_0 = 1$ and $\Lambda_1 = \mathbb{R}$. For $n \geq 1$, define

$$S_n = \frac{1}{nt}\frac{Z_{n-1}Z_{n+1}}{Z_n^2}.$$

We note that $S_n = \partial_{xy}\log Z_n$, from Lemma 3.6 below. For $z \in \Lambda_{n-1}$ and $y \in \Lambda_n$, write $z \prec y$ if $y_1 \geq z_1 > y_2 \geq \cdots > y_{n-1} \geq z_{n-1} > y_n$. For $y \in \Lambda_n^\circ$, denote by $GT(y)$ the Gelfand-Tsetlin polytope

$$\{(y^1, y^2, \ldots, y^{n-1}) \in \Lambda_1 \times \Lambda_2 \times \cdots \times \Lambda_{n-1} : \quad y^1 \prec y^2 \prec \cdots \prec y^{n-1} \prec y\}.$$

Theorem 3.4. *For $t > 0$, $x \in \mathbb{R}$ and $y \in \Lambda_n^\circ$,*

$$\hat{Z}_n(t, \hat{x}, y) = \Delta(y)^{-1}\prod_{i=1}^n Z(t, x, y_i)\int_{GT(y)}\prod_{k=1}^{n-1}\prod_{i=1}^{n-k} S_k(t, x, y_i^{n-k})dy_i^{n-k}.$$

In the case $\phi = 0$, this reduces to the fact that the volume of $GT(y)$ is proportional to $\Delta(y)$. By (16) and Proposition 3.1, this theorem can be seen as a consequence of the next two lemmas and the fact that $c_{n-1,t}c_{n+1,t}/c_{n,t}^2 = nt$,

Lemma 3.5. *If $f_1, f_2, \ldots$ is a sequence of continuously differentiable functions on $\mathbb{R}$ with $f_1 \equiv 1$ then*

$$\det[f_i(y_j)]_{i,j=1}^n = \int_{z\prec y}\det[f'_{i+1}(z_j)]_{i,j=1}^{n-1}dz_1\ldots dz_{n-1}.$$

Proof. Using the formula

$$1_{z\prec y} = \det\left[1_{y_{j+1}<z_i\leq y_j}\right]_{i,j=1}^{n-1}$$

we have, by Cauchy-Binet,

$$\begin{aligned}
&\int_{z\prec y}\det[f'_{i+1}(z_j)]_{i,j=1}^{n-1}dz_1\ldots dz_{n-1} \\
&\qquad = \int_{\Lambda_{n-1}}\det[f'_{i+1}(z_j)]_{i,j=1}^{n-1}\det\left[1_{y_{j+1}<z_i\leq y_j}\right]_{i,j=1}^{n-1}dz_1\ldots dz_{n-1} \\
&\qquad = \det\left[\int_{y_{j+1}}^{y_j} f'_{i+1}(z)dz\right]_{i,j=1}^{n-1} = \det\left[f_{i+1}(y_j) - f_{i+1}(y_{j+1})\right]_{i,j=1}^{n-1} \\
&\qquad = \det[f_i(y_j)]_{i,j=1}^n,
\end{aligned}$$

as required. □

Lemma 3.6. *Let $g(x, y)$ be a smooth function and define $W_0 = 1$, $W_1 = g$ and, for $n \geq 2$, $W_n = \det\left[\partial_x^i\partial_y^j g(x, y)\right]_{i,j=0}^{n-1}$. Suppose that W_n is strictly positive for all $n \geq 1$ and define $T_n = W_{n-1}W_{n+1}/W_n^2$. Then the following identities hold:*

$$T_n = \partial_{xy}\log W_n = \partial_y(\partial_y(\cdots\partial_y(\partial_x^n g/g)/T_1)/T_2)\cdots)/T_{n-1}),$$

$$\det[\partial_x^{i-1}g(x, y_j)]_{i,j=1}^n = \prod_{i=1}^n g(x, y_i)\int_{GT(y)}\prod_{k=1}^{n-1}\prod_{i=1}^{n-k} T_k(x, y_i^{n-k})dy_i^{n-k}.$$

Proof. From well-known properties of Wronskian determinants, $\partial_x W_n$, $\partial_y W_n$ and $\partial_{xy} W_n$ can be expressed as determinants, namely

$$\partial_x W_n = \det\left[\partial_x^i \partial_y^j g(x,y)\right]_{i=0,1,\ldots,n-2,n;\ j=0,\ldots,n-1},$$
$$\partial_y W_n = \det\left[\partial_x^i \partial_y^j g(x,y)\right]_{i=0,\ldots,n-1;\ j=0,1,\ldots,n-2,n},$$
$$\partial_{xy} W_n = \det\left[\partial_x^i \partial_y^j g(x,y)\right]_{i=0,1,\ldots,n-2,n;\ j=0,1,\ldots,n-2,n}.$$

It follows from Sylvester's determinant identity [**14**, p22] that

$$W_n \partial_{xy} W_n - (\partial_x W_n)(\partial_y W_n) = W_{n-1} W_{n+1},$$

proving the first identity. Similarly, for any $k \geq 1$,

$$W_n \partial_x^k \partial_y W_n - (\partial_x^k W_n)(\partial_y W_n) = W_{n-1} \partial_x^{k-1} W_{n+1}.$$

This implies that

$$(\partial_y(\partial_x^k W_n/W_n))/T_n = (\partial_x^{k-1} W_{n+1})/W_{n+1}.$$

In particular,

$$(\partial_y(\partial_x^n g/g))/T_1 = (\partial_x^{n-1} W_2)/W_2,$$
$$(\partial_y(\partial_x^{n-1} W_2/W_2))/T_2 = (\partial_x^{n-2} W_3)/W_3,$$

and so on, yielding the second identity.

Now, by Lemma 3.5,

$$\det\left[\frac{\partial_x^{i-1} g(x,y_j)}{g(x,y_j)}\right]_{i,j=1}^n = \int_{y^{n-1}\prec y} \det\left[\partial_{y_j^{n-1}} \frac{\partial_x^i g(x,y_j^{n-1})}{g(x,y_j^{n-1})}\right]_{i,j=1}^{n-1} \prod_{i=1}^{n-1} dy_i^{n-1}$$
$$= \int_{y^{n-1}\prec y} \det\left[\frac{\partial_{y_j^{n-1}} \frac{\partial_x^i g(x,y_j^{n-1})}{g(x,y_j^{n-1})}}{T_1(x,y_j^{n-1})}\right]_{i,j=1}^{n-1} \prod_{i=1}^{n-1} T_1(x,y_i^{n-1}) dy_i^{n-1}.$$

Applying Lemma 3.5 again, using $T_1 = \partial_y(\partial_x g/g)$, we obtain

$$\det\left[\frac{\partial_{y_j^{n-1}} \frac{\partial_x^i g(x,y_j^{n-1})}{g(x,y_j^{n-1})}}{T_1(x,y_j^{n-1})}\right]_{i,j=1}^{n-1}$$
$$= \int_{y^{n-2}\prec y^{n-1}} \det\left[\partial_{y_j^{n-2}} \frac{\partial_{y_j^{n-2}} \frac{\partial_x^{i+1} g(x,y_j^{n-2})}{g(x,y_j^{n-2})}}{T_1(x,y_j^{n-2})}\right]_{i,j=1}^{n-2} \prod_{i=1}^{n-2} dy_i^{n-2}$$
$$= \int_{y^{n-2}\prec y^{n-1}} \det\left[\frac{\partial_{y_j^{n-2}} \frac{\partial_{y_j^{n-2}} \frac{\partial_x^{i+1} g(x,y_j^{n-2})}{g(x,y_j^{n-2})}}{T_1(x,y_j^{n-2})}}{T_2(x,y_j^{n-2})}\right]_{i,j=1}^{n-2} \prod_{i=1}^{n-2} T_2(x,y_i^{n-2}) dy_i^{n-2}.$$

Now apply Lemma 3.5 again, using $T_2 = \partial_y(\partial_y(\partial_x^2 g/g)/T_1)$, and so on, to obtain the third identity. □

The evolution of the S_n's is given by the following proposition.

Proposition 3.7. *For $n \geq 1$,*

$$\partial_t S_n = \frac{1}{2}\partial_y^2 S_n + \partial_y[S_n \partial_y \log u_n]. \tag{17}$$

Proof. From Proposition 3.3 the functions $h_n = \log u_n$ satisfy

$$\partial_t h_n = \frac{1}{2}\partial_y^2 h_n + \frac{1}{2}(\partial_y h_n)^2 + \phi(t,y) + \partial_y^2 \log\left(Z_{n-1}/p^{n-1}\right).$$

It follows immediately that $S_1 = \partial_{xy} h_1$ satisfies

$$\partial_t S_1 = \frac{1}{2}\partial_y^2 S_1 + \partial_y[S_1 \partial_y h_1].$$

We prove the general statement by induction. Assume the induction hypothesis

$$\partial_t S_n = \frac{1}{2}\partial_y^2 S_n + \partial_y[S_n \partial_y h_n].$$

Now $S_{n+1} = S_n + \partial_{xy} h_{n+1}$ and

$$\partial_t h_{n+1} = \frac{1}{2}\partial_y^2 h_{n+1} + \frac{1}{2}(\partial_y h_{n+1})^2 + \phi(t,y) + \partial_y^2 \log\left(Z_n/p^n\right).$$

Thus,

$$\partial_t S_{n+1} = \frac{1}{2}\partial_y^2 S_{n+1} + \partial_y[S_n \partial_y h_n] + \partial_y[(S_{n+1} - S_n)\partial_y h_{n+1}] + \partial_y^2 S_n.$$

But

$$\partial_y^2 S_n = \partial_y[S_n \partial_y \log S_n] = \partial_y[S_n(\partial_y h_{n+1} - \partial_y h_n)],$$

so we have

$$\partial_t S_{n+1} = \frac{1}{2}\partial_y^2 S_{n+1} + \partial_y[S_{n+1}\partial_y h_{n+1}],$$

as required. □

4. The Karlin-McGregor formula

For $n = 1, 2, \ldots$ and $x, y \in \Lambda_n^\circ$, define

$$\tilde{\mathcal{Z}}_n(t,x,y) = p_n^*(t,x,y)\Big(1 + \sum_{k=1}^\infty \int_{\Delta_k(t)} \int_{\mathbb{R}^k} \tilde{R}_k^{(n)}((t_1,x_1),\ldots,(t_k,x_k)) \times W(dt_1,dx_1)\cdots W(dt_k,dx_k)\Big), \tag{18}$$

where $\tilde{R}_k^{(n)}$ is the k-point correlation function for a collection of n non-intersecting Brownian bridges started at positions x and ending at positions y at time t.

Proposition 4.1. *The series (18) is convergent in $L^2(W)$.*

Proof. We need to show that $\mathbb{E}[e^L 1_A] < \infty$ where L is the total intersection local time between two independent sets of n independent Brownian bridges started at positions x and ending at positions y at time t, and A is the event that each set is non-intersecting. So it suffices to show that $\mathbb{E}e^L < \infty$. By considering pairwise intersection local times and applying Hölder's inequality one obtains $\mathbb{E}e^L \leq \mathbb{E}e^{n^2\sqrt{t/2}R}$ where R is the local time at zero of a standard Brownian bridge on $[0,1]$, which has the Rayleigh distribution $P(R > r) = e^{-r^2/2}$, $r > 0$ (see, for example, [**24**]). □

For each n, the function $\tilde{\mathcal{Z}}_n(t,x,y)$ satisfies the equation

$$\partial_t \tilde{\mathcal{Z}}_n = \frac{1}{2}\Delta_y \tilde{\mathcal{Z}}_n + \sum_i \dot{W}(t,y_i)\tilde{\mathcal{Z}}_n,$$

with initial condition $\tilde{\mathcal{Z}}_n(0,x,y) = \prod_i \delta(x_i - y_i)$ and Dirichlet boundary conditions on $\partial\Lambda_n$. By this we mean that $\tilde{\mathcal{Z}}_n(t,x,y)$ satisfies the integral equation
(19)

$$\tilde{\mathcal{Z}}_n(t,x,y) = p_n^*(t,x,y) + \sum_i \int_0^t \int_{\Lambda_n} p_n^*(t-s,y',y)\tilde{\mathcal{Z}}_n(s,x,y')W(ds,dy_i')\prod_{j\neq i} dy_j'.$$

Iterating the integral equation (19) and then integrating out the y_j' variables which don't appear in the white noise yields the chaos expansion (18). Moreover, for each $x \in \Lambda_n^\circ$, $\tilde{\mathcal{Z}}_n(t,x,\cdot)$, $t \geq 0$ is a Markov process, which can be seen as a consequence of the 'flow property'

$$\tilde{\mathcal{Z}}_n(s+t,x,y) = \int_{\Lambda_n} \tilde{\mathcal{Z}}_n(s,x,z)\tilde{\mathcal{Z}}_n(t,z,y;s),$$

where $\tilde{\mathcal{Z}}_n(t,x,y;s)$ is defined via the chaos expansion (18) but with the shifted white noise $\dot{W}(s+\cdot,\cdot)$. This flow property can be seen directly from the chaos expansion or by taking S-transforms of both sides and using independence.

Theorem 4.2. *For $x,y \in \Lambda_n^\circ$, $\tilde{\mathcal{Z}}_n(t,x,y) = \det[\mathcal{Z}(t,x_i,y_j)]_{i,j=1}^n$.*

Proof. Let $\phi \in E$, multiply both sides by $\exp^\diamond W(\phi)$ and take expectations. The LHS becomes $\tilde{Z}_n(t,x,y)$, defined earlier by (11), which satisfies

$$\partial_t \tilde{Z}_n = \frac{1}{2}\Delta_y \tilde{Z}_n + \sum_i \phi(t,y_i)\tilde{Z}_n,$$

with initial condition $\tilde{Z}_n(0,x,y) = \prod_i \delta(x_i - y_i)$ and Dirichlet boundary conditions $\tilde{Z}_n(t,x,y) = 0$ for $t > 0$ and $y \in \partial\Lambda_n$. The RHS becomes

$$C_n(t,x,y) := \mathbb{E}\left[\det[\mathcal{Z}(t,x_i,y_j)]_{i,j=1}^n \exp^\diamond(W(\phi))\right],$$

which is given as follows. Let $\beta^1,\dots,\beta^n$ be a collection of independent Brownian bridges which start at positions x and end at positions y at time t. Define

$$V_n(t,x,y) = p_n(t,x,y)\mathbb{E}\exp\left(\sum_{i=1}^n \int_0^t \phi(s,\beta_s^i) + L_t\right),$$

where $p_n(t,x,y) = \prod_i p(t,x_i,y_i)$ and

$$L_t = \sum_{i<j} \int_0^t \delta(\beta_s^i - \beta_s^j)ds$$

denotes the total intersection local time between the Brownian bridges. Note that $V_n(t,x,y)$ is defined for any $x,y \in \mathbb{R}^n$, and satisfies

$$V_n(t,x,y) = \mathbb{E}\left(\prod_{i=1}^n \mathcal{Z}(t,x_i,y_i)\exp^\diamond(W(\phi))\right).$$

This can be seen by multiplying together the chaos expansions of each term in the product and taking expectations, which can be justified by the fact [**5**] that each term is in $L^p(W)$ for all $p \geq 1$. It follows that

$$C_n(t,x,y) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) V_n(t,\sigma x, y). \tag{20}$$

Note that $C_n(0,x,y) = \prod_i \delta(x_i - y_i)$ and, for $t > 0$, $C_n(t,x,y) = 0$ for $y \in \partial\Lambda_n$.

Now, by Feyman-Kac, for each $\sigma \in S_n$, $V_n(t,\sigma x,y)$ satisfies the equation

$$\partial_t u = \frac{1}{2}\Delta_y u + \sum_i \phi(t,y_i)u + \sum_{i<j} \delta(y_i - y_j)u,$$

in its integral form; more precisely,

$$\begin{aligned} V_n(t,\sigma x,y) = p_n(t,\sigma x,y) &+ \sum_i \int_0^t \int_{\mathbb{R}^n} p_n(t-s,y',y) V_n(s,\sigma x,y')\phi(s,y_i')dy'ds \\ &+ \sum_{i<j} \int_0^t \int_{\mathbb{R}^n} p_n(t-s,y',y) V_n(s,\sigma x,y')\delta(y_i' - y_j')dy'ds. \end{aligned}$$

By linearity, $C_n(t,x,y)$ (defined by (20) for $y \in \mathbb{R}^n$) also solves this integral equation and by the Dirichlet boundary conditions satisfied by $C_n(t,x,y)$ we obtain

$$\begin{aligned} C_n(t,x,y) = p_n(t,x,y) &+ \sum_i \int_0^t \int_{\mathbb{R}^n} p_n(t-s,y',y) C_n(s,x,y')\phi(s,y_i')dy'ds \\ &+ \sum_{i<j} \int_0^t \int_{\mathbb{R}^n} p_n(t-s,y',y) C_n(s,x,y')\delta(y_i' - y_j')dy'ds \\ &= p_n(t,x,y) + \sum_i \int_0^t \int_{\mathbb{R}^n} p_n(t-s,y',y) C_n(s,x,y')\phi(s,y_i')dy'ds. \end{aligned}$$

It follows that $C_n(t,x,y)$ satisfies

$$\partial_t C_n = \frac{1}{2}\Delta_y C_n + \sum_i \phi(t,y_i)C_n,$$

with initial condition $C_n(0,x,y) = \prod_i \delta(x_i - y_i)$ and Dirichlet boundary conditions on $\partial\Lambda_n$.

Now we will argue that $V_n(t,x,y)$, and hence $C_n(t,x,y)$, vanishes as $|y| \to \infty$ uniformly for t in any compact interval. Indeed, since ϕ is bounded, we have for $t \in [0,T]$ say,

$$|V_n(t,x,y)| \leq p_n(t,x,y)e^{CnT}\mathbb{E}e^{L_t},$$

where C is a constant. It is straightforward to obtain a bound on $\mathbb{E}\exp(L_t)$ which is uniform in $t \in [0,T]$ and $x, y \in \mathbb{R}^n$: by considering pairwise intersection local times and applying Hölder's inequality one obtains

$$\mathbb{E}e^{L_t} \leq \mathbb{E}e^{n(n-1)\sqrt{T/8}R},$$

where R is the local time at zero of a standard Brownian bridge on $[0,1]$. Thus $V_n(t,x,y) \to 0$ as $|y| \to \infty$ uniformly for $t \in [0,T]$, as required.

By uniqueness, we conclude that $C_n(t,x,y) = \tilde{Z}_n(t,x,y)$ for $t > 0$ and $x, y \in \Lambda_n^\circ$. Since this holds for any $\phi \in E$, we are done. $\square$

5. On the evolution of the partition functions

In this section we discuss the analogue of Theorem 3.4 in the white noise setting, and the implication that $(\mathcal{Z}_1(t,x,\cdot),\dots,\mathcal{Z}_n(t,x,\cdot)),\ t\ge 0$ is Markov.

We expect, but will not prove here, that for each $t>0$,

$$\hat{\mathcal{Z}}_n(t,x,y)=\frac{\tilde{\mathcal{Z}}_n(t,x,y)}{\Delta(x)\Delta(y)}$$

has a version which almost surely extends continuously to a strictly positive function on $\Lambda_n\times\Lambda_n$. In particular, for each $t>0$, almost surely,

$$\mathcal{Z}_n(t,a,b)=c_{n,t}\lim_{x\to\hat{a},y\to\hat{b}}\hat{\mathcal{Z}}_n(t,x,y), \tag{21}$$

uniformly on compact intervals. Assuming this continuity it can be shown that the analogue of Theorem 3.4 holds in the white-noise setting, that is, if we set $\mathcal{Z}_0=1$ and define, for $n\ge 1$,

$$\mathcal{S}_n=\frac{1}{nt}\frac{\mathcal{Z}_{n-1}\mathcal{Z}_{n+1}}{\mathcal{Z}_n^2},$$

then, for $t>0$, $x\in\mathbb{R}$ and $y\in\Lambda_n^\circ$,

$$\hat{\mathcal{Z}}_n(t,\hat{x},y)=\Delta(y)^{-1}\prod_{i=1}^n\mathcal{Z}_i(t,x,y_i)\int_{GT(y)}\prod_{k=1}^{n-1}\prod_{i=1}^{n-k}\mathcal{S}_k(t,x,y_i^{n-k})dy_i^{n-k}. \tag{22}$$

It is not difficult to see that for each $x\in\mathbb{R}$ and for each n, the process $\hat{\mathcal{Z}}_n(t,\hat{x},\cdot)$, $t\ge 0$ has the Markov property. Assuming the validity the formulas (21) and (22) this would imply that $(\mathcal{Z}_1(t,x,\cdot),\dots,\mathcal{Z}_n(t,x,\cdot)),\ t\ge 0$ is a Markov process.

A proof of the existence of an almost surely continuous extension for $\hat{\mathcal{Z}}_n(t,x,y)$ based on Kolmogorov's criterion would be long and technical. Here we will satisfy ourselves with a continuous extension in L_2, which then allows us to prove (22), and hence the Markov property of the multi-layer process, in the special case $n=2$.

Lemma 5.1. *For each $t>0$,*

$$\hat{\mathcal{Z}}_n(t,x,y)=\frac{\tilde{\mathcal{Z}}_n(t,x,y)}{\Delta(x)\Delta(y)}$$

extends continuously in $L_2(W)$ to $\Lambda_n\times\Lambda_n$. Moreover this extension satisfies

$$\mathcal{Z}_n(t,x,y)=c_{n,t}\hat{\mathcal{Z}}_n(t,\hat{x},\hat{y}).$$

Proof. First we recall that we have the representation

$$\hat{\mathcal{Z}}_n(t,x,y)=\frac{p_n^*(t,x,y)}{\Delta(x)\Delta(y)}\Big(1+\sum_{k=1}^\infty\int_{\Delta_k(t)}\int_{\mathbb{R}^k}\tilde{R}_k^{(n)}((t_1,x_1),\dots,(t_k,x_k)) \times W(dt_1,dx_1)\cdots W(dt_k,dx_k)\Big), \tag{23}$$

where $\tilde{R}_k^{(n)}$ are the correlations functions of a collection of n non-inersecting Brownian bridges starting at x and ending at time t at y. Since

$$\frac{p_n^*(t,x,y)}{\Delta(x)\Delta(y)}$$

extends continuously to $\Lambda_n \times \Lambda_n$ this representation naturally defines the extension of $\hat{\mathcal{Z}}$ to $\Lambda_n \times \Lambda_n$. Our task is show continuity in $L_2(W)$. For this it is enough to show that

$$(x, y, x', y') \mapsto \mathbb{E}\big[\hat{\mathcal{Z}}_n(t, x, y)\hat{\mathcal{Z}}_n(t, x', y')\big]$$

is continuous. Now, similarly to as in Theorem 3.4 this expectation is equal to

$$\left(\frac{p_n^*(t, x, y)}{\Delta(x)\Delta(y)}\right)^2 \mathbb{E}[e^L]$$

where L is the total intersection local time of two independent sets of non-intersection bridges, X and X' say.

Let us write $L = L_{[0,\delta]} + L_{[\delta,t-\delta]} + L_{[t-\delta,t]}$, where $L_{[0,\epsilon]}$ denotes the local time accrued over the times periods $[0, \delta]$ and so on. By conditioning on the position of the bridges at times δ and $t - \delta$ we have

$$\mathbb{E}\big[\exp(L_{[\delta,t-\delta]})|(X(0), X'(0), X(t), X'(t)) = z\big] = \\ \int p(z, \zeta)\mathbb{E}\big[\exp(L_{[\delta,t-\delta]})|(X(\delta), X'(\delta), X(t - \delta), X'(t - \delta)) = \zeta\big]d\zeta.$$

The kernel $p(z, \zeta)$ can be written as a product of transition densities for non-intersecting Brownian motions, and is thus seen to be continuous. From this it follows by a dominated convergence argument that

$$\mathbb{E}\big[\exp(L_{[\delta,t-\delta]})|(X(0), X'(0), X(t), X'(t)) = z\big]$$

depends continuously on z also.

To deduce the continuity of $z \mapsto \mathbb{E}\big[\exp(L)|(X(0), X'(0), X(t), X'(t)) = z\big]$ we must show that the difference

$$\mathbb{E}\big[\exp(L)|(X(0), X'(0), X(t), X'(t)) = z\big] \\ - \mathbb{E}\big[\exp(L_{[\delta,t-\delta]})|(X(0), X'(0), X(t), X'(t)) = z\big]$$

is can be made uniformly small for z within compact sets by choosing δ small enough. Applying the Cauchy-Schwartz inequality this amounts to showing that

$$\mathbb{E}\big[\exp(4L_{[0,\delta]})|(X(0), X'(0), X(t), X'(t)) = z\big]$$

and

$$\mathbb{E}\big[\exp(L_{[t-\delta,t]})|(X(0), X'(0), X(t), X'(t)) = z\big]$$

can be made uniformly close to 1, and this can be achieved using similar calculations to those made in Theorem 3.4. □

Theorem 5.2. *For $x \in \mathbb{R}$ and $y \in \Lambda_2^\circ$,*

$$\frac{\hat{Z}_2(t, \hat{x}, y)}{\mathcal{Z}(x, y_1)\mathcal{Z}(x, y_2)} = \frac{1}{y_1 - y_2}\int_{y_2}^{y_1} \frac{\hat{\mathcal{Z}}_2(t, \hat{x}, \hat{z})}{\mathcal{Z}(t, x, z)^2}dz. \tag{24}$$

PROOF. It is known [**5**] that the solution to the stochastic equation $\mathcal{Z}(t, x, y)$ admits a version that is almost surely continuous in t and y and moreover strictly positive. We assume in the following that we are using this version. In particular, having fixed t, x and $y_1 > y_2$ we let $A_\epsilon(x)$ be the event $\{\mathcal{Z}(t, x, z) > \epsilon$ for all $z \in [y_2, y_1 + 1]\}$. Then as $\epsilon \downarrow 0$ we have $\mathbb{P}(A_\epsilon(x)) \uparrow 1$.

By Theorem 4.2, for $u, v \in \Lambda_2^\circ$,

$$\hat{\mathcal{Z}}_2(t,u,v) = \frac{1}{\Delta(u)\Delta(v)}[\mathcal{Z}(t,u_1,v_1)\mathcal{Z}(t,u_2,v_2) - \mathcal{Z}(t,u_1,v_2)\mathcal{Z}(t,u_2,v_1)].$$

Hence,

$$\frac{\hat{\mathcal{Z}}_2(t,u,v)}{\mathcal{Z}(t,u_2,v_1)\mathcal{Z}(t,u_2,v_2)} = \frac{1}{\Delta(u)\Delta(v)}\left[\frac{\mathcal{Z}(t,u_1,v_1)}{\mathcal{Z}(t,u_2,v_1)} - \frac{\mathcal{Z}(t,u_1,v_2)}{\mathcal{Z}(t,u_2,v_2)}\right]. \tag{25}$$

Let $h > 0$, $v = (z+h, z)$ and integrate this equation with respect to z over the interval $[y_2, y_1]$ thus obtaining the identity

$$\begin{aligned}&\int_{y_2}^{y_1} \frac{\hat{\mathcal{Z}}_2(t,u,(z+h,z))}{\mathcal{Z}(t,u_2,z+h)\mathcal{Z}(t,u_2,z)}dz\\ &\quad = \frac{1}{(u_1-u_2)h}\left[\int_{y_1}^{y_1+h}\frac{\mathcal{Z}(t,u_1,z)}{\mathcal{Z}(t,u_2,z)}dz - \int_{y_2}^{y_2+h}\frac{\mathcal{Z}(t,u_1,z)}{\mathcal{Z}(t,u_2,z)}dz\right].\end{aligned}$$

Now let h tend to zero. By the continuity of $\mathcal{Z}(u_1,\cdot)$ and $\mathcal{Z}(u_2,\cdot)$ the RHS converges almost surely to

$$\frac{1}{(u_1-u_2)}\left[\frac{\mathcal{Z}(t,u_1,y_1)}{\mathcal{Z}(t,u_2,y_1)} - \frac{\mathcal{Z}(t,u_1,y_2)}{\mathcal{Z}(t,u_2,y_2)}\right]$$

We want to identify the limit of the LHS. Consider

$$\begin{aligned}E &= \left|\int_{y_2}^{y_1}\frac{\hat{\mathcal{Z}}_2(t,u,(z+h,z))}{\mathcal{Z}(t,u_2,z+h)\mathcal{Z}(t,u_2,z)}dz - \int_{y_2}^{y_1}\frac{\hat{\mathcal{Z}}_2(t,u,\hat{z})}{\mathcal{Z}(t,u_2,z)^2}dz\right|\\ &\le \int_{y_2}^{y_1}\frac{|\hat{\mathcal{Z}}_2(t,u,(z+h,z)) - \hat{\mathcal{Z}}_2(t,u,\hat{z})|}{\mathcal{Z}(t,u_2,z)^2}dz\\ &\quad + \int_{y_2}^{y_1}\frac{|\mathcal{Z}(t,u_2,z+h)-\mathcal{Z}(t,u_2,z)|\hat{\mathcal{Z}}_2(t,u,(z+h,z))}{\mathcal{Z}(t,u_2,z)^2\mathcal{Z}(t,u_2,z+h)}dz\end{aligned}$$

We have

$$\begin{aligned}\mathbb{E}\big[E;A_\epsilon(u_2)\big] &\le \epsilon^{-2}\int_{y_2}^{y_1}\mathbb{E}|\hat{\mathcal{Z}}_2(t,u,(z+h,z)) - \hat{\mathcal{Z}}_2(t,u,\hat{z})|dz\\ &\quad + \epsilon^{-3}\int_{y_2}^{y_1}\big(\mathbb{E}[\hat{\mathcal{Z}}_2(t,u,(z+h,z))^2]\mathbb{E}[(\mathcal{Z}(t,u_2,z+h)-\mathcal{Z}(t,u_2,z))^2]\big)^{1/2}\big)dz\end{aligned} \tag{26}$$

By virtue of the uniform continuity in L_2 of the mappings $(z_1,z_2)\mapsto\hat{\mathcal{Z}}_2(t,u,(z_1,z_2))$ and $z\mapsto\mathcal{Z}(t,u_1,z)$ these integrals tends to zero as $h\downarrow 0$, and consequently E tends to 0 in probability. Thus we have proven

$$\frac{1}{(u_1-u_2)}\left[\frac{\mathcal{Z}(t,u_1,y_1)}{\mathcal{Z}(t,u_2,y_1)} - \frac{\mathcal{Z}(t,u_1,y_2)}{\mathcal{Z}(t,u_2,y_2)}\right] = \int_{y_2}^{y_1}\frac{\hat{\mathcal{Z}}_2(t,u,\hat{z}))}{\mathcal{Z}(t,u_2,z)^2}dz. \tag{27}$$

Next let $u = (x+h,x)$ and let $h\downarrow 0$. The LHS of (27) can be rewritten as

$$(y_1-y_2)\frac{\hat{\mathcal{Z}}(t,(x+h,x),(y_1,y_2))}{\mathcal{Z}(t,x+h,y_1)\mathcal{Z}(t,x,y_2)};$$

as $h\downarrow 0$ this converges in probability to

$$(y_1-y_2)\frac{\hat{\mathcal{Z}}(t,\hat{x},(y_1,y_2))}{\mathcal{Z}(t,x,y_1)\mathcal{Z}(t,x,y_2)}.$$

On the other hand, if we consider

$$F = \left| \int_{y_2}^{y_1} \frac{\hat{\mathcal{Z}}_2(t,(x+h,x),\hat{z}))}{\mathcal{Z}(t,x,z)^2} dz - \int_{y_2}^{y_1} \frac{\hat{\mathcal{Z}}_2(t,\hat{x},\hat{z}))}{\mathcal{Z}(t,x,z)^2} dz \right|$$

we have

$$\mathbb{E}\big[F; A_\epsilon(x)\big] \le \epsilon^{-2} \int_{y_2}^{y_1} \mathbb{E}|\hat{\mathcal{Z}}_2(t,(x+h,x),\hat{z}) - \hat{\mathcal{Z}}_2(t,\hat{x},\hat{z})| dz$$

which again by the L_2 continuity of $\hat{\mathcal{Z}}_2$ converges to 0 as $h \downarrow 0$. From this it follows the RHS of (27) converges to

$$\int_{y_2}^{y_1} \frac{\hat{\mathcal{Z}}_2(t,\hat{x},\hat{z}))}{\mathcal{Z}(t,x,z)^2} dz,$$

and the result is proven. □

We remark that the identity (27) shows that the ratio of two solutions to the stochastic heat equation is in H^1; in fact, it has recently been shown by Hairer [**13**] to be in $C^{3/2-\epsilon}$.

Corollary 5.3. *For $x \in \mathbb{R}$, $(\mathcal{Z}_1(t,x,\cdot), \mathcal{Z}_2(t,x,\cdot)),\ t \ge 0$ is a Markov process.*

It will also be interesting to understand the evolution of the multi-layer process in terms of a system of SPDEs. Motivated by the evolution equations obtained in Section 3 in the the case of a smooth potential, it is natural to consider and to try to make sense of the system of equations

$$\partial_t \mathcal{S}_n = \frac{1}{2}\partial_y^2 \mathcal{S}_n + \partial_y[\mathcal{S}_n \partial_y \log \mathcal{U}_n], \tag{28}$$

where $\mathcal{Z}_n = \mathcal{U}_1 \cdots \mathcal{U}_n$. For recent progress in this direction, see [**13**].

6. Concluding remarks

As remarked in the introduction, the multi-layer construction presented in this paper is based on the tropical RSK correspondence, so it is natural to consider such an interpretation in the continuum setting. The analogue of RSK in the context of section 3 is the mapping $\phi\big|_{[0,t]\times\mathbb{R}} \mapsto \{u_n(t,0,\cdot),\ n \ge 1\}$. In the language of RSK, $\{u_n(t,0,x),\ n \ge 1;\ x \ge 0\}$ is the P-tableau, $\{u_n(t,0,-x),\ n \ge 1;\ x \ge 0\}$ is the Q-tableau, and their common 'shape' is the sequence $\{u_n(t,0,0),\ n \ge 1\}$. We note the following symmetry, which corresponds to a well known symmetry property of the RSK correspondence. Writing $f = \phi\big|_{[0,t]\times\mathbb{R}}$, $P(f) = \{u_n(t,0,x),\ n \ge 1;\ x \ge 0\}$, $Q(f) = \{u_n(t,0,-x),\ n \ge 1;\ x \ge 0\}$ and $f^\dagger(s,x) = f(s,-x)$, we have: $P(f^\dagger) = Q(f)$ and $Q(f^\dagger) = P(f)$.

Similarly, in the white noise setting, we define

$$P(W^t) = \{\mathcal{U}_n(t,0,x),\ n \ge 1;\ x \ge 0\} \qquad Q(W^t) = \{\mathcal{U}_n(t,0,-x),\ n \ge 1;\ x \ge 0\}$$

where W^t denotes the restriction of W to $[0,t] \times \mathbb{R}$. Given the main result of the paper [**22**], it is natural to expect that $P(W^t)$ and $Q(W^t)$ are diffusion processes in $\mathbb{R}^{\mathbb{N}}$ (indexed by $x \ge 0$) which are conditionally independent given their starting position $\{\mathcal{U}_n(t,0,0),\ n \ge 1\}$. This would be the analogue, in this setting, of

Pitman's '$2M - X$' theorem. In recent work, Corwin and Hammond [**8**] have shown that the process $\{\mathcal{U}_n(t,0,x),\ n \geq 1;\ x \in \mathbb{R}\}$ has a kind of Gibbs property and, moreover, has all components strictly positive almost surely; they call the logarithm of this process the 'KPZ line ensemble'. For large t, it should rescale to the multi-layer Airy process (or 'Airy line ensemble') which was introduced by Prahofer and Spohn [**25**]. At present, we only know this to be the case for the one-point distributions of the first layer: it has been shown in the papers [**2, 31**] that the distribution of $\log[\mathcal{Z}(t,0,x)/p(t,0,x)]$ (which is independent of x) converges in a suitable scaling limit to the Tracy-Widom distribution. This result on the first layer has been tentatively extended to the finite-dimenisonal distributions by Prolhac and Spohn [**26, 27**].

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